PH 712 Probability and Statistical Inference

Part II: Transformation Between RVs

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Find the pmf of $Y = g(X)$, given the pmf of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. Calculate $p_Y(y) = \Pr(Y = y) = \Pr(X \in \{x \in \text{supp}(X) : y = g(x)\})$

Example Lec2.1

Let X have the pmf $p_X(x) = 2^x \mathbf{1}_{\{-1,-2,\ldots\}}(x)$. Find the pmf of $Y = X^2$.

Find the cdf of $Y = g(X)$, given the distribution of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. Calculate $F_Y(y) = \Pr\{g(X) \le y\} = \Pr(X \in \{x \in \text{supp}(X) : g(x) \le y\})$

Example Lec2.2

Suppose $X \sim U([-\pi/2, \pi/2])$, i.e., its pdf is $f_X(x) = \pi^{-1}\mathbf{1}_{[-\pi/2,\pi/2]}(x)$. Find the cdf of $Y = X^2$.

Find the pdf of $Y = g(X)$, given the pdf of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. $$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} \int_{\{x:g(x)\le y\}} f_X(x)\mathrm{d}x$$
   - Integration region $\{x : g(x) \le y\}$ may be expressed in terms of a series of intervals with endpoints as functions of y, say $[a(y), b(y)]$, $[c(y), d(y)]$, etc.
   - The integration of f_X at the far-right end is often avoidable by employing the Leibniz Rule (CB Thm. 2.4.1):
     $$\frac{\mathrm{d}}{\mathrm{d}y} \int_{a(y)}^{b(y)} f(x)\mathrm{d}x = f\{b(y)\}\frac{\mathrm{d}}{\mathrm{d}y} b(y) - f\{a(y)\}\frac{\mathrm{d}}{\mathrm{d}y} a(y)$$
     with $a(y)$ and $b(y)$ both differentiable with respect to y.

Example Lec2.2'

Let X have the uniform pdf $f_X(x) = \pi^{-1}\mathbf{1}_{[-\pi/2,\pi/2]}(x)$. Find the pdf of $Y = X^2$.

Example Lec2.3

$X \sim \text{Weibull}(\text{shape} = \alpha > 0, \text{scale} = \beta > 0)$, i.e., $f_X(x) = (\alpha/\beta)(x/\beta)^{\alpha-1}\exp\{-(x/\beta)^\alpha\}\mathbf{1}_{(0,\infty)}(x)$. Find the pdf of $Y = \ln X$.

Example Lec2.4

Let X have the pdf $f_X(x) = 2^{-1}\mathbf{1}_{(0,2)}(x)$. Find the pdf of $Y = X^2$.

Example Lec2.5

Suppose $f_X(x) = 3^{-1}\mathbf{1}_{(-1,2)}(x)$. Find the pdf of $Y = X^2$.

Example Lec2.6

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$, i.e., $f_X(x) = (\sigma\sqrt{2\pi})^{-1}\exp\{-(x-\mu)^2/(2\sigma^2)\}$. Find the pdf of $Y = aX + b$ with $a \neq 0$.