PH 712 Probability and Statistical Inference

Part VI: Evaluating Estimators I

Zhiyang Zhou (zhou67@uwm.edu, zhiyanggeezhou.github.io)

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Bias

- Bias of $\hat{\theta}$: $\text{Bias}(\hat{\theta}) = \text{E}(\hat{\theta}) - \theta$
- Unbiased: if $\text{E}(\hat{\theta}) = \theta$

Mean squared error (MSE)

- $\text{MSE}(\hat{\theta}) = \text{E}(\hat{\theta} - \theta)^2 = \text{Bias}^2(\hat{\theta}) + \text{var}(\hat{\theta})$
  - The lower the better
- For unbiased estimators, minimizing the MSE $\Leftrightarrow$ minimizing the variance

Cramér-Rao lower bound (CRLB, CB Thm 7.3.9 & Lemma 7.3.11)

- Recall the score $S(\theta) = \ell'(\theta)$
- CRLB for the variance of any unbiased estimator of $g(\theta)$: $I_n^{-1}(\theta)\left\{\frac{\text{d}}{\text{d}\theta}g(\theta)\right\}^2$
  - I.e., under certain conditions, for any statistic T_n such that $\text{E}(T_n) = g(\theta)$, $\text{var}(T_n) \geq I_n^{-1}(\theta)\left\{\frac{\text{d}}{\text{d}\theta}g(\theta)\right\}^2$
    - The right-hand side reducing to $I_n^{-1}(\theta)$ if $g(\theta) = \theta$
    - $I_n(\theta) = \text{var}\{S(\theta)\} = \text{E}[\{S(\theta)\}^2] = -\text{E}\{H(\theta)\}$ called the Fisher information
      - The most convenient way to calculate $I_n(\theta)$: $I_n(\theta) = -\text{E}\{H(\theta)\}$
    - $H(\theta) = S'(\theta) = \ell''(\theta)$ called the Hessian

Example Lec6.1

- Find the CRLB for all the UNBIASED estimators in the following cases.
  a. $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with UNKNOWN μ and GIVEN σ^2.
  b. $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with GIVEN μ and UNKNOWN σ^2.

Efficiency (HMC Def 6.2.2)

- For an UNBIASED estimator of $g(\theta)$, say T_n, the *efficiency* of T_n is the ratio of the CRLB to $\text{var}(T_n)$, i.e., $\text{CRLB}/\text{var}(T_n)$.
  - The higher efficiency the better (typically up to 1);
  - T_n is an efficient estimator for $g(\theta) \iff \text{E}(T_n) = g(\theta)$ and its efficiency = 1.