PH 712 Probability and Statistical Inference

Part III: Normal sampling theory

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Random samples, realizations and statistics

- RVs $X_1, \ldots, X_n$: a random sample of size n
  - Independent and identially distributed (iid) sample: $X_1, \ldots, X_n$ are iid
  - iid normal sample: $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$
- Realization: the actual observed value or outcome of an RV
  - Denoted by lowercase letters
  - E.g.,
    * x_i is the realization of X_i if $X_i = x_i$ is observed
    * Observed $X_1 = x_1, \ldots, X_n = x_n \Rightarrow x_1, \ldots, x_n$ is the realization of $X_1, \ldots, X_n$
- Statistic: any function of a random sample, e.g.,
  - Sample mean: $\bar{X} = n^{-1} \sum_{i=1}^n X_i$
  - Sample variance: $S^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X})^2$
  - Sample standard deviation: $S = \sqrt{(n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X})^2}$

Defining χ^2-, t-, and F-RVs in terms of an iid normal sample (HMC Chp. 3)

- $\sum_{i=1}^n X_i^2 \sim \chi^2(n)$ if $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$
  - $Q \sim \chi^2(n) \Rightarrow \mathrm{E}(Q) = n$ and $\mathrm{var}(Q) = 2n$
- $Z/\sqrt{Q/n} \sim t(n)$ if $Z \sim \mathcal{N}(0, 1)$ and $Q \sim \chi^2(n)$ are independent of each other
- $(P/m)/(Q/n) \sim F(m, n)$ if $P \sim \chi^2(m)$ and $Q \sim \chi^2(n)$ are independent of each other

More identities for an iid normal sample

- $n^{1/2}(\bar{X} - \mu)/\sigma \sim \mathcal{N}(0, 1)$
- $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$
- $\bar{X} \perp S^2$
- $n^{1/2}(\bar{X} - \mu)/S \sim t(n-1)$