PH 712 Probability and Statistical Inference

Part X: Recap

Zhiyang Zhou (zhou67@uwm.edu, zhiyanggeezhou.github.io)

2025/12/08 00:56:37

Workflow of statistical inference (making decision based on data)

1. Prior to statistical inference
    - Research gap
    - Research question
    - Research hypotheses
2. Collect data (realizations of RVs of your interest)
3. Assume a statistical model for data
4. Translate research hypotheses into statistical terms
5. Hypothesis testing
6. Make decision

Statistical model

- Characterizing distributions
    - cdf, pdf, pmf
- Checking independence
    - Separable joint cdf: $F_{X,Y}(x, y) = F_X(x)F_Y(y)$
    - Separable joint pdf or pmf: $f_{X,Y}(x, y) = f_X(x)f_Y(y)$
    - Conditional pdf or pmf: $f_{X|Y}(x \mid y) = f_X(x)$

Point estimation

- MLE
    - Maximizing the likelihood or log-likelihood with respect to $\boldsymbol{\theta} \in \boldsymbol{\Theta}$
    - Properties
        * Invariance: $\widehat{g(\boldsymbol{\theta})}_{\mathrm{ML}} = g(\hat{\boldsymbol{\theta}}_{\mathrm{ML}})$
        * Consistency: $\tau(\hat{\theta}_{\mathrm{ML}}) \approx \tau(\theta)$
        * Asymptotic distribution (by delta methods)
            · $\tau'(\theta) \neq 0 \Rightarrow \sqrt{n}\{\tau(\hat{\theta}_{\mathrm{ML}}) - \tau(\theta)\} \approx N(0, \{\tau'(\theta)\}^2/I_1(\theta))$.
- Evaluating estimators
    - MSE
        * For unbiased estimators: MSE = var $\geq$ CRLB
    - Consistency: $\hat{\theta}_n \xrightarrow{p} \theta$
    - Asymptotic efficiency: $\sqrt{n}\{\tau(\hat{\theta}_n) - \tau(\theta)\} \xrightarrow{d} \mathcal{N}(0, \{\tau'(\theta)\}^2/I_1(\theta))$

Hypothesis testing

- $H_0 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_0$ vs. $H_1 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_1$.

- $\mathbf{\Theta} = \mathbf{\Theta}_0 \cup \mathbf{\Theta}_1$
  - $\emptyset = \mathbf{\Theta}_0 \cap \mathbf{\Theta}_1$
- LRT (equivalent to the UMP test when the UMP test exists)
  - Test statistic

$$\lambda(\boldsymbol{x}) = \frac{\sup_{\theta \in \mathbf{\Theta}_0} L(\theta)}{\sup_{\boldsymbol{\theta} \in \mathbf{\Theta}} L(\theta)} = \frac{L(\hat{\theta}_{0,\mathrm{ML}})}{L(\hat{\theta}_{\mathrm{ML}})}$$

  - Critical value c_α satisfying

$$\sup_{\theta \in \Theta_0} \Pr\{\lambda(\boldsymbol{X}) \le c_\alpha \mid \theta\} = \alpha$$

    * Asymptotically, $c_\alpha \approx \exp(-\chi^2_{\nu,1-\alpha}/2)$
  - Reject H_0 if $\lambda(\boldsymbol{x}) \le c_\alpha$
- Wald test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \ne \theta_0$
  - Test statistic $(\hat{\theta}_{\mathrm{ML}} - \theta_0)/\sqrt{\widehat{\mathrm{var}}(\hat{\theta}_{\mathrm{ML}})}$
    * $\widehat{\mathrm{var}}(\hat{\theta}_{\mathrm{ML}})$ obtained by CRLB/bootstrap
  - Reject H_0 if $|\hat{\theta}_{\mathrm{ML}} - \theta_0|/\sqrt{\widehat{\mathrm{var}}(\hat{\theta}_{\mathrm{ML}})} \ge \Phi^{-1}_{1-\alpha/2}$
  - Asymptotically equivalent to LRT for this two sided test
- p-value
  - Giving a standardized rejection region: reject H_0 if p-value $\le \alpha$

$(1-\alpha) \times 100\%$ confidence set of θ

- Inverting a level α rejection region for two-sided hypotheses
- Bootstrap